Gorenstein simplices with a given $\delta$-polynomial

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This is joint work with T. Hibi and K. Yoshida
\( \mathcal{P} \subset \mathbb{R}^d \): a lattice polytope of dimension \( d \).

We define \( \delta(\mathcal{P}, t) \) by the formula

\[
\delta(\mathcal{P}, t) = (1 - t)^{d+1} \left[ 1 + \sum_{n \geq 1} |\mathcal{P} \cap \mathbb{Z}^d| t^n \right].
\]

Then \( \delta(\mathcal{P}, t) \) is a polynomial in \( t \) of degree at most \( d \).

We call \( \delta(\mathcal{P}, t) \) the \( \delta \)-polynomial of \( \mathcal{P} \).
Some properties of $\delta$-polynomials

Set $\delta(\mathcal{P}, t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$.
Then

$> \delta_0 = 1$, $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^d| - (d + 1)$ and $\delta_d = |\text{int}(\mathcal{P}) \cap \mathbb{Z}^d|$  

Hence $\delta_1 \geq \delta_d$

$> \delta_i \in \mathbb{Z}_{\geq 0}$ for each $i$

$> \delta_0 + \cdots + \delta_d =$ the normalized volume of $\mathcal{P}$  
i.e., (the usual volume of $\mathcal{P}$) $\times d!$

$> \text{and more...}$
Classification problems

Problem

Characterize the polynomials with nonnegative integer coefficients that are the $\delta$-polynomials of some lattice polytopes.
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Remark
For a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$, the lattice pyramid is

$$\text{Pyr}(\mathcal{P}) = \text{conv}(\mathcal{P} \times \{0\}, (0, \ldots, 0, 1)) \subset \mathbb{R}^{d+1}$$

Then $\delta(\mathcal{P}, t) = \delta(\text{Pyr}(\mathcal{P}), t)$. 
A polynomial $a_0 + a_1 t + \cdots + a_s t^s \ (a_s \neq 0)$ is symmetric if $a_i = a_{s-i}$ for $i = 0, \ldots, s$.

**Definition**

$\mathcal{P}$ is **Gorenstein** if its $\delta$-polynomial is symmetric.

> In each dimension, there exist only finitely many Gorenstein polytopes.

> For fixed degree, there exist only finitely many Gorenstein polytopes that are not lattice pyramids.
Gorenstein simplices with prime volumes

**Proposition**

Let $p$ be a prime number and $\Delta$ a Gorenstein simplex with the normalized volume $p$. Then $\delta(\Delta, t)$ forms

$$\delta(\Delta, t) = 1 + t^k + \cdots + t^{(p-1)k}$$

with some positive integer $k$.  

Main Problem

Problem
Given positive integers $k$ and $v$, classify the Gorenstein simplices with the $\delta$-polynomial $1 + t + \cdots + t^{(v-1)k}$.

Remark
- For any $k$ and $v$, there exists a lattice simplex with the $\delta$-polynomial $1 + t + \cdots + t^{(v-1)k}$.
- In this problem, when $k > 1$, its target is Gorenstein empty simplices.
Associated abelian groups

For a lattice simplex $\Delta \subset \mathbb{R}^d$ of dimension $d$ whose vertices are $v_0, \ldots, v_d \in \mathbb{Z}^d$ set

$$\Lambda_\Delta = \{(\lambda_0, \ldots, \lambda_d) \in (\mathbb{R}/\mathbb{Z})^{d+1} : \sum_{i=0}^d \lambda_i(v_i, 1) \in \mathbb{Z}^{d+1}\}.$$ 

The collection $\Lambda_\Delta$ forms a finite abelian group with addition defined as follows:

$$+ : (\mathbb{R}/\mathbb{Z})^{d+1} \times (\mathbb{R}/\mathbb{Z})^{d+1} \to (\mathbb{R}/\mathbb{Z})^{d+1}$$

$$(\lambda_0, \ldots, \lambda_d) + (\lambda'_0, \ldots, \lambda'_d) = (\lambda_0 + \lambda'_0, \ldots, \lambda_d + \lambda'_d)$$
Conversely, let $\Lambda \subset (\mathbb{R}/\mathbb{Z})^{d+1}$ be a finite abelian subgroup of $(\mathbb{R}/\mathbb{Z})^{d+1}$ such that for any $(\lambda_0, \ldots, \lambda_d) \in \Lambda$, $\sum_{i=0}^{d} \lambda_i \in \mathbb{Z}$ (*). Then there exists a lattice $d$-simplex $\Delta$ such that $\Lambda = \Lambda_{\Delta}$.
Conversely, let $\Lambda \subset (\mathbb{R}/\mathbb{Z})^{d+1}$ be a finite abelian subgroup of $(\mathbb{R}/\mathbb{Z})^{d+1}$ such that for any $(\lambda_0, \ldots, \lambda_d) \in \Lambda$, $\sum_{i=0}^{d} \lambda_i \in \mathbb{Z}$ (*). Then there exists a lattice $d$-simplex $\Delta$ such that $\Lambda = \Lambda_\Delta$.

In particular,

$$\{\Delta : \text{lattice } d\text{-simplices}\} \overset{1:1}{\leftrightarrow} \{\Lambda : \text{finite abelian groups(*)}\}$$

$\sim_1$ : unimodular equivalence

$\sim_2$ : permutation of the coordinates
Computing $\delta$-polynomials

Let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension $d$ and
\[ \delta(\Delta, t) = \delta_0 + \cdots + \delta_d t^d. \]
Then each $\delta_i$, one has
\[ \delta_i = \#\{ (\lambda_0, \ldots, \lambda_d) \in \Lambda_\Delta : \sum_{j=0}^{d} \lambda_j = i \} \]

Lemma
\[ \Delta \text{ is a lattice pyramid if and only if there exists some index } i \text{ such that for any } (\lambda_0, \ldots, \lambda_d) \in \Lambda_\Delta, \lambda_i = 0. \]
The case \( \nu = p \)

Proposition

Let \( p \) be a prime number, \( k \) a positive integer and \( \Delta \) a Gorenstein simplex with the \( \delta \)-polynomial \( 1 + t^k + \cdots + t^{(p-1)k} \). Assume that \( \Delta \) is not a lattice pyramid. Then \( \Lambda_\Delta \) is generated by

\[
\left( \frac{1}{p}, \ldots, \frac{1}{p} \right) \in (\mathbb{R}/\mathbb{Z})^{pk}.
\]
The case $n = 3$ and $k = 1$

Example
There are two lattice non-simplices with the $\delta$-polynomial $1 + t + t^2$.

1. $\mathcal{P}_1 = \text{conv}(0, e_1, e_2, e_3, e_1 + e_2 + e_3) \subset \mathbb{R}^3$
2. $\mathcal{P}_2 = \text{conv}(0, e_1, e_2, e_3, e_4, e_1 + e_3, e_2 + e_4) \subset \mathbb{R}^4$
The case $v = p^2$

**Theorem**
Let $p$ be a prime number, $k$ a positive integer and $\Delta$ a Gorenstein simplex with the $\delta$-polynomial $1 + t^k + \cdots + t^{(p^2-1)k}$. Assume that $\Delta$ is not a lattice pyramid. Then a system of generators of the finite abelian group $\Lambda_\Delta$ is the set of row vectors of the matrix which can be written up to permutation of the columns as follows:

1. $\left(1/p^2 \cdots 1/p^2\right) \in \left(\mathbb{R}/\mathbb{Z}\right)^{1 \times p^2k}$;

2. $\begin{pmatrix} 1/p & \cdots & 1/p \\ \vdots & \ddots & \vdots \\ 1/p & \cdots & 1/p \end{pmatrix} \in \left(\mathbb{R}/\mathbb{Z}\right)^{1 \times (p^2+p-1)k}$;

3. $\begin{pmatrix} 1/p & \cdots & 1/p \\ 0 & \cdots & 0 \end{pmatrix} \in \left(\mathbb{R}/\mathbb{Z}\right)^{2 \times p(p+1)k}$. 
The case $v = 4$ and $k = 1$

Example

There exist 7 lattice non-simplices with the $\delta$-polynomial $1 + t + t^2 + t^3$.

1. $Q_1 = \text{conv}(0, e_1, e_2, e_3, e_1 + e_2 + 2e_4, e_1 - e_3) \subset \mathbb{R}^4$;
2. $Q_2 = \text{conv}(0, e_1, \ldots, e_4, e_1 + e_2 + 2e_5, e_3 + e_4) \subset \mathbb{R}^5$;
3. $Q_3 = \text{conv}(0, e_1, \ldots, e_5, e_1 + e_2 + 2e_6, -e_3 + e_4 + e_5) \subset \mathbb{R}^6$;
4. $Q_4 = \text{conv}(0, e_1, \ldots, e_4, e_1 + e_2 + e_3 + e_4) \subset \mathbb{R}^4$;
5. $Q_5 = \text{conv}(0, e_1, \ldots, e_4, -e_1 + -e_2 + 2e_3 + 2e_4) \subset \mathbb{R}^4$;
6. $Q_6 = \text{conv}(0, e_1, \ldots, e_5, -e_1 + e_2 + e_3 + e_4 + e_5) \subset \mathbb{R}^5$;
7. $Q_7 = \text{conv}(0, e_1, \ldots, e_6, -e_1 - e_2 + e_3 + e_4 + e_5 + e_6) \subset \mathbb{R}^6$. 
The case $\nu = pq$

**Theorem**

Let $p$ and $q$ be prime numbers with $p \neq q$, $k$ a positive integer and $\Delta$ a Gorenstein simplex with the $\delta$-polynomial $1 + t^k + \cdots + t^{(pq-1)k}$. Assume that $\Delta$ is not a lattice pyramid. Then the finite abelian group $\Lambda_{\Delta}$ is generated by one of the following elements which can be written up to permutation of the coordinates as follows:
The case \( \nu = pq \)

1. \( (1/(pq), \ldots, 1/(pq)) \in (\mathbb{R}/\mathbb{Z})^{pqk}; \)

2. \( \left(\frac{1}{p}, \ldots, \frac{1}{p}, \frac{1}{q}, \ldots, \frac{1}{q}\right) \quad \begin{cases} \text{pk} \\ \text{pqk} \end{cases} \in (\mathbb{R}/\mathbb{Z})^{p(q+1)k}; \)

3. \( \left(\frac{1}{q}, \ldots, \frac{1}{q}, \frac{1}{p}, \ldots, \frac{1}{p}\right) \quad \begin{cases} \text{qk} \\ \text{pqk} \end{cases} \in (\mathbb{R}/\mathbb{Z})^{(p+1)qk}; \)

4. \( \left(\frac{1}{q}, \ldots, \frac{1}{q}, \frac{1}{(pq)}, \ldots, \frac{1}{(pq)}\right) \quad \begin{cases} \text{(pq−1)k} \\ \text{pk} \end{cases} \in (\mathbb{R}/\mathbb{Z})^{(pq+p−1)k}; \)

5. \( \left(\frac{1}{p}, \ldots, \frac{1}{p}, \frac{1}{(pq)}, \ldots, \frac{1}{(pq)}\right) \quad \begin{cases} \text{(pq−1)k} \\ \text{qk} \end{cases} \in (\mathbb{R}/\mathbb{Z})^{(pq+q−1)k}. \)
Given integers $v \geq 1$ and $k \geq 1$, let $N(v, k)$ denote the number of Gorenstein simplices with the $\delta$-polynomial $1 + t^k + \cdots + t^{(v-1)k}$ up to unimodular equivalence and lattice pyramid constructions. Then each $k, p$ and $q$, one has

1. $N(p, k) = 1$;
2. $N(p^2, k) = 3$;
3. $N(pq, k) = 5$. 
The number of certain Gorenstein simplices

Conjecture

Let $p_1, \ldots, p_\ell$ be distinct prime numbers, $a_1, \ldots, a_\ell$ positive integers, $k$ a positive integer. Then

1. $N(p_1^{a_1} \cdots p_\ell^{a_\ell}, 1) = \cdots = N(p_1^{a_1} \cdots p_\ell^{a_\ell}, k) = \cdots$.

2. For any distinct prime numbers $q_1, \ldots, q_\ell$, 
   $N(p_1^{a_1} \cdots p_\ell^{a_\ell}, k) = N(q_1^{a_1} \cdots q_\ell^{a_\ell}, k)$.

Namely, for positive integers $v$ and $k$, $N(v, k)$ depends on only the divisor lattice of $v$. 
Join Construction

Let $P \subset \mathbb{R}^d$, $Q \subset \mathbb{R}^e$ be lattice polytopes. Then the join

$$P \star Q = \text{conv} \left( \{ (x, 0_e, 0), (0_d, y, 1) : x \in P, y \in Q \} \right) \subset \mathbb{R}^{d+e+1}$$

has the $\delta$-polynomial $\delta(P, t) \cdot \delta(Q, t)$.

Proposition

Given positive integers $v_1, v_2$ and $k$, let $\Delta_1$ and $\Delta_2$ be lattice simplices such that $\delta(\Delta_1, t) = 1 + t^k + \cdots + t^{(v_1-1)k}$ and $\delta(\Delta_2, t) = 1 + t^{v_1k} + \cdots + t^{v_1(v_2-1)k}$. Then one has

$$\delta(\Delta_1 \star \Delta_2, t) = 1 + t^k + \cdots + t^{(v_1v_2-1)k}.$$
Non-join simplex

Theorem

Given a positive integer \( \nu \), let \( \Delta \subset \mathbb{R}^d \) be a lattice simplex of dimension \( d \) such that \( \Lambda_\Delta \) is generated by

\[
\left( \frac{1}{\nu_1}, \ldots, \frac{1}{\nu_1}, \frac{1}{\nu_2}, \ldots, \frac{1}{\nu_2}, \ldots, \frac{1}{\nu_t}, \ldots, \frac{1}{\nu_t} \right) \in (\mathbb{R}/\mathbb{Z})^{d+1},
\]

where \( 1 = \nu_0 < \nu_1 < \cdots < \nu_t = \nu \) and for any \( 1 \leq i \leq t - 1 \), \( \nu_i \mid \nu_{i+1} \) and \( s_1, \ldots, s_t \) are positive integers. Then

\[
\delta(\Delta, t) = 1 + t^k + \cdots + t^{(\nu-1)k}
\]

with a nonnegative integer \( k \) if and only if

\[
s_i = \begin{cases} 
\left( \frac{V_t}{\nu_{i-1}} - \frac{V_t}{\nu_{i+1}} \right) k, & 1 \leq i \leq t - 1 \\
\frac{V_t}{\nu_{t-1}} k, & i = t,
\end{cases}
\]
Given positive integers \( v, k \), let \( M(v, k) \) denote the number of Gorenstein simplices, up to unimodular equivalence, which are appeared in the previous Theorem. Then one has

\[
M(v, k) = \sum_{n \in D_v \setminus \{v\}} M(n, k),
\]

where \( D_v \) is the set of divisors of \( v \).
The number of certain Gorenstein simplices

Corollary

Let $p_1, \ldots, p_\ell$ be distinct prime numbers, $a_1, \ldots, a_\ell$ positive integers, $k$ a positive integer. Then

1. $M(p_1^{a_1} \cdots p_\ell^{a_\ell}, 1) = \cdots = M(p_1^{a_1} \cdots p_\ell^{a_\ell}, k) = \cdots.$

2. For any distinct prime numbers $q_1, \ldots, q_\ell,$

   $M(p_1^{a_1} \cdots p_\ell^{a_\ell}, k) = M(q_1^{a_1} \cdots q_\ell^{a_\ell}, k).$

Namely, for positive integers $v$ and $k,$ $M(v, k)$ depends on only the divisor lattice of $v.$
Example

(1) Let \( v = p^\ell \) with a prime number \( p \) and a positive integer \( \ell \). Then one has \( M(v, k) = 2^{\ell-1} \).

(2) Let \( v = p_1 \cdots p_\ell \), where \( p_1, \ldots, p_\ell \) are distinct prime numbers. Set \( M(v, k) = a(\ell) \).

Then one has

\[
M(v, k) = a(\ell) = \sum_{i=0}^{\ell-1} \binom{\ell}{i} a(i).
\]

\( a(1), a(2), \ldots \) are
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1, 3, 13, 75, 541, 4683, 47293, 545835, 7087261, 102247563, 1622632573, 28091567595, 526858348381, 10641342970443, 230283190977853, 5315654681981355, 130370767029135901, 3385534663256845323, 92801587319328411133, 2677687796244384203115, \ldots