Pattern-Avoiding Polytopes

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Let $\mathfrak{S}_n$ denote the symmetric group on 1, 2, ..., $n$, $\pi \in \mathfrak{S}_k$ and $\sigma \in \mathfrak{S}_n$, written as words.

**Definition**

Say $\sigma$ contains the pattern $\pi$ if there is some substring of $\sigma$ whose elements have the same relative order as those in $\pi$. If no such substring exists, then $\sigma$ avoids the pattern $\pi$. If $\Pi \subseteq \mathfrak{S}$, then $\sigma$ avoids $\Pi$ if $\sigma$ avoids every element of $\Pi$.

So 526413 does not avoid 132 while 453621 does.

Denote by

$$\text{Av}_n(\Pi) := \{ \sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids } \Pi \}$$

the avoidance class of $\Pi$. 
Pattern Avoidance

A simple yet difficult question: given $\Pi$, determine $|\text{Av}_n(\Pi)|$.

If $\pi = a_1 a_2 \ldots a_k$, call

$$\pi^r := a_k a_{n-1} \ldots a_1$$

the reversal of $\pi$ and

$$\pi^c := (k - a_1 + 1)(k - a_2 + 1)\ldots(k - a_k + 1)$$

the complement of $\pi$. Then $|\text{Av}_n(\pi)| = |\text{Av}_n(\pi^r)| = |\text{Av}_n(\pi^c)|$.

Definition

Say $\pi_1$ and $\pi_2$ are **Wilf equivalent**, written $\pi_1 \equiv \pi_2$, if

$|\text{Av}_n(\pi_1)| = |\text{Av}_n(\pi_2)|$ for all $n$.

Wilf equivalence is an equivalence relation on $\mathfrak{S}$. 
Pattern Avoidance

So $\pi \equiv \pi^r \equiv \pi^c$. In fact, $\pi$ is Wilf equivalent to any permutation obtained by acting on its diagram by the dihedral group of the square. These are called the trivial Wilf equivalences.

Example

\[
\begin{align*}
4261573 & \equiv 4271536 & \equiv 4627315 & \equiv 2537164
\end{align*}
\]
Pattern Avoidance

Theorem (MacMahon (1915) and Knuth (1968))

If $\pi \in \mathfrak{S}_3$, then for all $n$, $|\text{Av}_n(\pi)| = C_n$, the $n^{th}$ Catalan number.

Theorem (Erdős-Szekeres (1935))

For any positive integers $a, b$, every permutation of length at least $(a - 1)(b - 1) + 1$ contains the patterns $123 \ldots a$ or $b(b - 1)(b - 2) \ldots 1$.

Theorem (Billey, Burdzy, and Sagan (2012))

For all $n$, $|\text{Av}_n(132, 312)| = 2^{n-1}$.
Why study pattern avoidance?

- Stack-sortable permutations
  - A permutation is stack-sortable if and only if it avoids 231 (Knuth, 1968)

- Permutation statistics
  - Almost all known Mahonian permutation statistics really belong to a class of 14 statistics, if the use of vincular patterns is allowed (Babson and Steingrímsson, 2000)

- Classifying smooth / factorial / Gorenstein Schubert varieties using bivincular patterns (Úlfarsson, 2010)
Ehrhart Theory

Definition

For a lattice polytope $P \subseteq \mathbb{R}^n$, its **Ehrhart polynomial** is

$$\mathcal{L}_P(m) := |mP \cap \mathbb{R}^n|,$$

and its **Ehrhart series** is

$$E_P(t) := \sum_{m \geq 0} \mathcal{L}_P(m)t^m$$

$$= \frac{h^*_P(t)}{(1 - t)^{\dim P + 1}}.$$

The numerator $h^*_P(t)$ is the **$h^*$-polynomial** of $P$ and its list of coefficients $h^*(P) := (h_0^*, \ldots, h_d^*)$ is the **$h^*$-vector** of $P$. 
Two Big Questions

1. When is $h^*(P)$ palindromic?
   - This happens exactly when $P$ is Gorenstein, a property that is often reasonably detectable if a hyperplane description of $P$ is known.

2. When is $h^*(P)$ unimodal? Various sufficient conditions are known, but necessary conditions are not as clear.
Π-avoiding Permutahedra

**Definition**

The **permutohedron** is defined as

\[ P_n := \text{conv}\{(a_1, \ldots, a_n) \in \mathbb{R}^n | a_1 \ldots a_n \in \mathcal{S}_n\}. \]

Some quick facts about \( P_n \):

1. invariant under the action of \( \mathcal{S}_n \)
2. simple zonotope
3. its Ehrhart polynomial is

\[ \mathcal{L}_{P_n}(m) = \sum_{i=0}^{n-1} f_i^n m^i, \]

where \( f_i^n \) is the number of labeled forests on \( n \) vertices with \( i \) edges.
Π-avoiding Permutohedra

**Definition**

For $\Pi \subseteq \mathcal{S}$, define

$$P_n(\Pi) := \text{conv}\{ (a_1, \ldots, a_n) \mid a_1 \ldots a_n \in \text{Av}_n(\Pi) \}$$

to be the *Π-avoiding permutohedron*.

So if $\Pi = \emptyset$, then $P_n(\Pi) = P_n$.

Important note: this is not a subclass of generalized permutohedra introduced by Postnikov. This fact can be verified by comparing normal fans and using a theorem of Postnikov, Reiner, and Williams.
Π-avoiding Permutohedra

$P_n(\pi)$ is unimodularly equivalent to both $P_n(\pi^r)$ and $P_n(\pi^c)$. But that’s about where it stops.

**Example (Trivial Wilf equivalence $\not\Rightarrow$ unimodular equivalence)**

Choose $\pi = 1423$ and $\pi' = 2431$. These are related by a 90-degree rotation, but $P_5(\pi)$ has 48 facets while $P_5(\pi')$ only has 46.
Σ-avoiding Permutohedra

Theorem (D. and Sagan)

If \( \Pi = \{132, 312\} \), then \( P_n(\Pi) \) is a rectangular parallelepiped with Ehrhart polynomial

\[
\sum_{i=0}^{n-1} \frac{(n - 1)!}{(n - i - 1)!} m^i
\]

This extends the previous result \(| \text{Av}_n(132, 312) | = 2^{n-1} \).

Corollary

The number of interior lattice points of \( P_n(132, 312) \) is the number of derangements of \( \mathfrak{S}_{n-1} \).

(Follows from Ehrhart-Macdonald reciprocity)
Π-avoiding Permutohedra

Theorem (Beck, Jochemko, McCullough, in preparation)

Every lattice zonotope has a unimodal $h^*$-vector.

Corollary

For all $n$, $h^*(P_n(132, 312))$ is unimodal.
Theorem (D. and Sagan)

If $\Pi = \{123, 132\}$, then $P_n(\Pi)$ is a combinatorial (but not geometric!) cube with Ehrhart polynomial

\[
\frac{m+1}{(n-1)!} \prod_{j=2}^{n-1} (nm+j)
\]

($P_n(\Pi)$ is a Pitman-Stanley polytope)
Proposition (D. and Sagan)

If $\Pi = \{123, 132, 312\}$, then $P_n(\Pi)$ is a simplex with Ehrhart polynomial $(1 + m)^{n-1}$. Hence $h_P^*(t)$ is the Eulerian polynomial $A_{n-1}(t)$.

$P_n(123, 132, 312)$ is (unimodularly equivalent to) the simplex containing certain lecture hall partitions. Work of Corteel, Lee, and Savage imply the Ehrhart-theoretic results (an observation made by Ben Braun).
Π-avoiding Permutohedra

The results for the different avoidance classes were proven in very different ways.

This is common in the world of pattern avoidance.
Π-avoiding Birkhoff Polytopes

Definition

The $n \times n$ Birkhoff polytope is

$$B_n := \text{conv}\{M \in \mathbb{R}^{n \times n} \mid M \text{ a matrix for some } \sigma \in \mathfrak{S}_n\}$$

Some variations:

1. transportation polytopes
2. permutation polytopes (Burggraf, De Loera, Omar)
3. the “symmetric slice” of $B_n$ (Stanley, Jia)
Definition

For $\Pi \subseteq S$, define

$$B_n(\Pi) := \text{conv}\{M \in \mathbb{R}^{n \times n} \mid M \text{ a matrix for some } \sigma \in \text{Av}_n(\Pi)\}$$

to be the $\Pi$-avoiding Birkhoff polytope.

This time, if $\pi \in S_k$ and $\pi'$ are trivially Wilf equivalent, then $B_n(\pi)$ and $B_n(\pi')$ are unimodularly equivalent.
Alternating permutations

**Definition**

A permutation $a_1a_2\ldots a_n \in \mathfrak{S}_n$ is *alternating* if $a_1 < a_2 > a_3 < a_4 > a_5 < \ldots$.

Let $\widetilde{\text{Av}}_n(\Pi)$ denote the alternating permutations in $\mathfrak{S}_n$ that avoid $\Pi$. Analogously define $\widetilde{B}_n(\Pi)$.

These could also be described as $B_n(\Pi)$ for an appropriate $\Pi$ if we allow vincular patterns.

Our focus will be on the specific polytopes $B_n(132,312)$ and $\widetilde{B}_n(123)$. 
Π-avoiding Birkhoff Polytopes

Proposition (D. and Sagan)

For all $n$,

$$\dim B_n(132, 312) = \binom{n}{2}$$

and

$$\dim \tilde{B}_n(123) = \binom{\lceil n/2 \rceil}{2}$$

Beyond knowing the number of vertices of each, the combinatorial structures of these are completely unknown.
Theorem (Stanley (1970s), Athanasiadis (2005))

For all $n$, $h^*(B_n)$ is palindromic and unimodal.

What can we say about $h^*(B_n(\Pi))$?
Main Conjecture

Conjecture (D. and Sagan)

The $h^*$-vectors of $B_n(132, 312)$ and $\tilde{B}_n(123)$ are palindromic and unimodal.

Broad strategy:

1. Show that these polytopes have regular, unimodular triangulations
2. Show that these polytopes are Gorenstein
The posets $Q_n(\Pi)$ and $\tilde{Q}_n(\Pi)$

**Definition**

The right weak (Bruhat) order on $\mathfrak{S}_n$ is defined as $\sigma < \sigma'$ if $\sigma' = \sigma s_i$ for some simple transposition $s_i$ and $\sigma'$ has more inversions than $\sigma$. The left weak (Bruhat) order is defined analogously.

Let $Q_n(132, 312)$ be the poset on $\mathcal{A}v_n(132, 312)$ induced from the right weak order on $\mathfrak{S}_n$, and $\tilde{Q}_n(123)$ to be the poset on $\tilde{\mathcal{A}}v_n(123)$ induced from the left weak order on $\mathfrak{S}_n$. 
Examples: $Q_5(132, 312)$ and $\tilde{Q}_8(123)$
The posets $Q_n(\Pi)$ and $\tilde{Q}_n(\Pi)$

Theorem (D. and Sagan)

The following isomorphisms hold:

$$Q_n(132, 312) \cong M(n - 1),$$

where $M(k)$ is the lattice of shifted Young diagrams with largest part at most $k$, and

$$\tilde{Q}_n(123) \cong D^*[n/2],$$

where $D_k$ is the lattice of Dyck paths of length $2k$ such that if $d_1, d_2 \in D_k$, then $d_1 < d_2$ if $d_1$ lies entirely underneath $d_2$. 
From here, we want to use the following facts:

- distributive lattices have EL-labelings
- posets with EL-labelings have shellable order complexes
- given a lattice polytope with a shellable unimodular triangulation, its $h^*$-vector can be computed based on information about the shelling order

Goal: show that the order complexes of $Q_n(132, 312)$ and $\tilde{Q}_n(123)$ induce shellable unimodular triangulations of $B_n(132, 312)$ and $\tilde{B}_n(123)$.
The Commutative Algebra

Conjecture (D. and Sagan)

$B_n(132, 312)$ and $\tilde{B}_n(123)$ have flag, regular unimodular triangulations.

Theorem (Sturmfels)

For a lattice polytope $P$, the initial ideals of the toric ideal $I_P$ are in bijection with the regular triangulations of $P$. The initial ideal of $I_P$ is squarefree if and only if the corresponding triangulation of $P$ is unimodular.
Watermelons, Stars, and Fermi Configurations

Definition

A watermelon $\overline{W}_{l,k}$ is the digraph with vertices

$$\{(i, j) \in \mathbb{Z}^2 \mid 0 \leq i \leq l, 0 \leq j \leq k, j \leq i\}$$

with an arc from $a$ to $b$ if $b - a \in \{-e_1, -e_2\}$. A star graph $S_n$ is the graph whose vertex set is

$$\{(i, j) \in \mathbb{Z}^2 \mid i, j \geq 0, i + j \leq n\}$$

with arcs formed the same way as with watermelons.

To make later definitions simpler, we introduce a unique sink $v$ for $S_n$ by including an arc from the points $(-i, -n + i)$ to $v$. 
Examples: $\overline{W}_{4,3}$ and $S_3$
**Definition**

A **Fermi configuration** in a digraph $H$ with source $u$ and sink $v$ is a collection of distinct, noncrossing paths from $u$ to $v$. A Fermi configuration is **maximal** if no additional distinct noncrossing paths from $u$ to $v$ can be included in the configuration.

**Example (A maximal Fermi configuration in $\overline{W}_{3,2}$)**

![Diagram of a maximal Fermi configuration]
Watermelons, Stars, and Fermi Configurations

**Definition**

A triple of adjacent paths in a maximal Fermi configuration is called a **flipflop** if the two 2-dimensional faces it bounds share no edges of the central path. If the central path goes to the right of the first 2-dimensional face it encounters, then the path is called **flopped**. Otherwise, it is **flipped**.

**Example**

The configuration on the previous slide contains the flopped walk \((p_2, p_3, p_4)\) but no flipped walks.
Theorem (Arrowsmith, Bhatti, and Essam (2012))

Suppose $H$ is a digraph with unique source and sink, and that $H$ has a unique minimal-cardinality Fermi configuration covering all of its arcs. Let $\varphi_k(H)$ denote the number of maximal Fermi configurations in $H$ that contain $k$ flopped walks. Then the polynomial

$$\Phi(H; t) = \sum_{i \geq 0} \varphi_k(H)t^k$$

has palindromic coefficients.
Sagan and I have shown that if the previously-mentioned conjecture holds, then $\Phi(S_n; t)$ is the $h^*$-polynomial for $B_n(132, 312)$ and $\Phi(\overline{W}_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}; t)$ is the $h^*$-polynomial for $\tilde{B}_n(123)$.

It appears that the coefficients of $\Phi(\overline{W}_{k,m}; t)$ are unimodal for all $k$ and $m$, but it is not immediately obvious how to choose $\Pi$ so that $\Phi(\overline{W}_{k,m}; t) = B_n(\Pi)$ (or if any such $\Pi$ exists)
Open Questions

1. Is there a nice combinatorial proof for the number of interior lattice points of $P_n(132, 312)$?

2. For “nice” special classes of $\Pi$,
   - what is the combinatorial structure of $P_n(\Pi)$ or $B_n(\Pi)$?
   - what is $\text{Vol}(P_n(\Pi)), \text{Vol}(B_n(\Pi))$?
   - what is the Ehrhart polynomial for $P_n(\Pi)$?
   - what is the $h^*$-vector of $B_n(\Pi)$?

3. What happens if we consider classes of vincular or bivincular patterns?

4. For which choices of $\Pi$ is $B_n(\Pi)$ IDP? Gorenstein?

5. What are the homotopy types of $Q_n(\Pi)$? (in general their order complexes aren’t necessarily spheres, or even Cohen-Macaulay)