Critical exponent for the Cauchy problem to the weakly coupled damped wave system

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Asymptotic Analysis for Nonlinear Dispersive and Wave Equations
In honor of Professor Nakao Hayashi’s 60th birthday
September 10, 2014

Joint work with Kenji Nishihara (Waseda University)
1 Introduction

2 Main results

3 Idea of the proof
Weakly coupled system

Weakly coupled system of damped wave equations

\[
\begin{align*}
(u_{tt} - \Delta u + u_t &= |v|^p, \\
v_{tt} - \Delta v + v_t &= |u|^q, \\
(u, u_t, v, v_t)(0, x) &= \varepsilon(u_0, u_1, v_0, v_1)(x).
\end{align*}
\]

- \( u = u(t, x), v = v(t, x) \): real-valued unknown functions,
- \( t \in (0, \infty), x \in \mathbb{R}^N, N \geq 1, \)
- \( p, q > 1, \)
- \( (u_0, u_1, v_0, v_1) \in [H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)]^2 : \text{compactly supported}, \)
- \( \varepsilon > 0. \)
Weakly coupled system of damped wave equations

\[
(\text{DW}) \quad \begin{cases}
  u_{tt} - \Delta u + u_t = |v|^p, \\
  v_{tt} - \Delta v + v_t = |u|^q,
\end{cases}
\]

\[
(u, u_t, v, v_t)(0, x) = \varepsilon(u_0, u_1, v_0, v_1)(x).
\]

Goal: to prove that

\[
\alpha := \max \left\{ \frac{p + 1}{pq - 1}, \frac{q + 1}{pq - 1} \right\} = \frac{N}{2}
\]

is critical for any dimension \( N \geq 1 \). The word “critical” means

\[
\alpha < \frac{N}{2} \Rightarrow \text{Small data global existence (SDGE)};
\]

\[
\alpha \geq \frac{N}{2} \Rightarrow \text{Blow-up in finite time}.
\]
Related results (1) : single semilinear problem

**Diffusion phenomenon**

The solution of the damped wave equation

\[ u_{tt} - \Delta u + u_t = 0 \]

behaves like that of the heat equation

\[ v_t - \Delta v = 0 \]

as \( t \to +\infty \).

For the semilinear damped wave equation

\[ u_{tt} - \Delta u + u_t = |u|^p, \]

Todorova-Yordanov (2001), Qi S. Zhang (2001) proved that \( p = \rho_F(N) := 1 + 2/N \) is critical, namely,

\[ p > \rho_F(N) \Rightarrow \text{SDGE}; \]
\[ 1 < p \leq \rho_F(N) \Rightarrow \text{Blow-up}. \]

For the system of heat equations

\[
\begin{align*}
  u_t - \Delta u &= |v|^p, \\
  v_t - \Delta v &= |u|^q,
\end{align*}
\]

Escobedo-Herrero (1991) proved that \( \alpha = \max\left\{ \frac{p+1}{pq-1}, \frac{q+1}{pq-1} \right\} = \frac{N}{2} \) is critical.

For (DW):

\[
\begin{align*}
  u_{tt} - \Delta u + u_t &= |v|^p, \\
  v_{tt} - \Delta v + v_t &= |u|^q,
\end{align*}
\]

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Takeda (2009), Ogawa-Takeda (2010, 2011): SDGE and asymptotic profile for General strongly coupled systems \( (N \leq 3) \).
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Main results (1)

\[
\begin{align*}
    u_{tt} - \Delta u + u_t &= |v|^p, \\
    v_{tt} - \Delta v + v_t &= |u|^q, \\
    (u, u_t, v, v_t)(0, x) &= \varepsilon(u_0, u_1, v_0, v_1)(x),
\end{align*}
\]

where \((u_0, u_1, v_0, v_1) \in \left[H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)\right]^2\) : compactly supported.

Let

\[X(T) := C([0, T); H^1(\mathbb{R}^N)) \times C^1([0, T); L^2(\mathbb{R}^N)).\]

Theorem 1

If \(1 < p \leq q < \infty (N = 1, 2), 1 < p \leq q \leq \frac{N}{N - 2} (N \geq 3),\) and

\[\alpha = \frac{q + 1}{pq - 1} < \frac{N}{2},\]

then \(\exists \varepsilon_0 > 0\) s.t. \(\forall \varepsilon \in (0, \varepsilon_0], \exists! (u, v) \in X(\infty)^2 :\) solution to (DW).
Let

\[ T_\varepsilon = \sup \{ T \in (0, \infty) \mid \exists! (u, v) \in X(T)^2 : \text{sol. to (DW)} \} \].

**Theorem 2**

If \( 1 < p \leq q < \infty \) (\( N = 1, 2 \)), \( 1 < p \leq q \leq \frac{N}{N-2} \) (\( N \geq 3 \)),

\[ \alpha = \frac{q + 1}{pq - 1} > \frac{N}{2}, \]

and \( \int_{\mathbb{R}^N} (u_0(x) + u_1(x))dx > 0 \), \( \int_{\mathbb{R}^N} (v_0(x) + v_1(x))dx > 0 \), then

\[ \exists C > 0 \text{ s.t. } \forall \varepsilon \in (0, 1], \]

\[ T_\varepsilon \leq C\varepsilon^{-1/\kappa}, \]

where

\[ \kappa = \alpha - \frac{N}{2} = \frac{q + 1}{pq - 1} - \frac{N}{2}. \]
1 Introduction

2 Main results

3 Idea of the proof
Idea of the proof: global existence part

Observation for the optimal decay rate (Nishihara (2012))

+ Weighted energy method (Todorova-Yordanov (2001))
Idea of the proof: global existence part

\[ q(p - \frac{2}{N}) = \rho_F(N) \]

\[ \alpha = \frac{q + 1}{pq - 1} < \frac{N}{2} \iff q(p - \frac{2}{N}) > \rho_F(N) = 1 + \frac{2}{N}. \]

Therefore, it suffices to consider the case where

\[ \max \left\{ 1, \frac{2}{N} \right\} < p \leq \rho_F(N) < q. \]
Idea of the proof: global existence part

WLOG we may assume $1 < p \leq q$

Note that

$$\alpha = \frac{q + 1}{pq - 1} < \frac{N}{2} \iff q(p - \frac{2}{N}) > \rho_F(N) = 1 + \frac{2}{N}.$$

Therefore, it suffices to consider the case where

$$\max \left\{ 1, \frac{2}{N} \right\} < p \leq \rho_F(N) < q.$$
Idea of the proof: global existence part

Note that \[ \alpha = \frac{q + 1}{pq - 1} < \frac{N}{2} \iff q(p - \frac{2}{N}) > \rho_F(N) = 1 + \frac{2}{N}. \]

Therefore, it suffices to consider the case where

\[ \max \left\{ 1, \frac{2}{N} \right\} < p \leq \rho_F(N) < q. \]
The diagram illustrates the function $\rho_F(N)$ and the inequality $q(p - \frac{2}{N}) = \rho_F(N)$. Note that $\alpha = \frac{q + 1}{pq - 1} < \frac{N}{2} \iff q(p - \frac{2}{N}) > \rho_F(N) = 1 + \frac{2}{N}$.

Therefore, it suffices to consider the case where

$$\max \left\{ 1, \frac{2}{N} \right\} < p \leq \rho_F(N) < q.$$
Let us assume
\[
\max \left\{ 1, \frac{2}{N} \right\} < p < \rho_F(N) < q
\]
and consider the integral equation
\[
u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} |v(s)|^p \, ds.
\]
Now we assume
\[
\|v(t)\|_{L^r} \leq C(1 + t)^{-\frac{N}{2} \left(1 - \frac{1}{r}\right)} \quad (1 \leq r \leq \infty).
\]
Then we see that
\[
\|u(t)\|_{L^r} \leq C(1 + t)^{-\frac{N}{2} \left(p - \frac{2}{N} - \frac{1}{r}\right)} \quad (1 \leq r \leq \infty),
\]
namely, \(\|u(t)\|_{L^1}\) may grow up and \(\|u(t)\|_{L^\infty}\) decays to 0.
In particular, we expect that
\[
\|u(t)\|_{L^2}^2 \leq C(1 + t)^{-N(p - \frac{2}{N} - \frac{1}{2})},
\]
\[
\|v(t)\|_{L^2}^2 \leq C(1 + t)^{-\frac{N}{2}}.
\]

Taking this into account, we define the weighted energy of \((u, v)\) by
\[
W_u(t) = (1 + t)^{N(p - \frac{2}{N} - \frac{1}{2}) + 1 - \delta(p + 1)} \left( \|e^{\psi(t, \cdot)} u_t(t)\|_{L^2}^2 + \|e^{\psi(t, \cdot)} \nabla u(t)\|_{L^2}^2 \right)
\]
\[
+ (1 + t)^{N(p - \frac{2}{N} - \frac{1}{2}) - \delta(p + 1)} \|e^{\psi(t, \cdot)} u(t)\|_{L^2}^2,
\]
\[
W_v(t) = (1 + t)^{\frac{N}{2} + 1 - \delta} \left( \|e^{\psi(t, \cdot)} v_t(t)\|_{L^2}^2 + \|e^{\psi(t, \cdot)} \nabla v(t)\|_{L^2}^2 \right)
\]
\[
+ (1 + t)^{\frac{N}{2} - \delta} \|e^{\psi(t, \cdot)} v(t)\|_{L^2}^2,
\]
where \(\delta > 0\),
\[
\psi(t, x) = \frac{|x|^2}{4(2 + \lambda)(1 + t)}, \quad \lambda > 0,
\]
Idea of the proof: weighted energy method (2)

Let

\[ M(t) = \sup_{s \in [0,t]} (W_u(s) + W_v(s)). \]

Multiplying (DW) by \( e^{2\psi} u_t \) and \( e^{2\psi} u \), respectively, we have

\[
\frac{1}{2} \frac{d}{dt} \int e^{2\psi} (u_t^2 + |\nabla u|^2) dx + \int e^{2\psi} \left( 1 + (-\psi_t) + \frac{|
abla \psi|^2}{-\psi_t} \right) u_t^2 dx
\]

\[
+ \int \frac{e^{2\psi}}{-\psi} \left| \psi_t \nabla u - u_t \nabla \psi \right|^2 dx = \int e^{2\psi} |v|^p u_t dx,
\]

\[
\frac{d}{dt} \int e^{2\psi} (uu_t + \frac{u^2}{2}) dx + \int e^{2\psi} ((-\psi_t) + 2|\nabla \psi|^2 + \Delta \psi) u^2 dx
\]

\[
+ \int e^{2\psi} \left( |\nabla u + 2u\nabla \psi|^2 - 2\psi_t uu_t - u_t^2 \right) dx = \int e^{2\psi} |v|^p u dx.
\]

The positive terms give the decay of weighted energy and \( L^2 \)-norm.

From them, we obtain the a priori estimate

\[ M(t) \leq C \varepsilon^2 + C(M(t)^p + M(t)^{(p+1)/2} + M(t)^q + M(t)^{(q+1)/2}). \]
General weakly coupled systems (cf. Takeda (2009, $N \leq 3$))

$$
\begin{align*}
(\partial_t^2 - \Delta + \partial_t)u_1 &= |u_k|^{p_1}, \\
(\partial_t^2 - \Delta + \partial_t)u_2 &= |u_1|^{p_2}, \\
&\, \quad \quad \quad \vdots \\
(\partial_t^2 - \Delta + \partial_t)u_k &= |u_{k-1}|^{p_k}.
\end{align*}
$$

How to observe the optimal decay rates?


$$
\begin{align*}
u_{tt} - \Delta u + u_t &= |u|^{p_{11}}|v|^{p_{12}}, \\
v_{tt} - \Delta v + v_t &= |u|^{p_{21}}|v|^{p_{22}}.
\end{align*}
$$

How to prove the blow-up of solutions?